COVER PAGE

Algebra Qualifying Examination

Wednesday, January 2, 2019

2:30pm – 4:30pm

C-304 Wells Hall

Your Sign-Up Number: ______________________

Note: Attach this cover page to the paperwork you are submitting to be graded. **This number should be the only identification appearing on all of your paperwork – DO NOT WRITE YOUR NAME on any of the paperwork you are submitting.**

Algebra Qualifying Exam I (January 2019)

1. (10 points) Let G be a group of order $175 = 5^2 \cdot 7$. Prove that G is an abelian group.

2. (15 points) Let p be a prime number and let $\mathbf{F}_p$ be the finite field of order p. Let $GL_2(\mathbf{F}_p)$ be the group of invertible 2×2 matrices over $\mathbf{F}_p$.
   (a) Determine the order of the group $GL_2(\mathbf{F}_p)$.
   (b) Prove that the unipotent subgroup
   $$U = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| \, a \in \mathbf{F}_p \right\}.$$
   is a p-Sylow subgroup of $GL_2(\mathbf{F}_p)$.
   (c) Determine the number n_p of p-Sylow subgroups in $GL_2(\mathbf{F}_p)$.

3. (15 points) Consider the ring
   $$\mathbb{Z}[\sqrt{3}] = \left\{ a + b\sqrt{3} \,\middle|\, a, b \in \mathbb{Z} \right\}.$$
   Define $N(a + b\sqrt{3}) = |a^2 - 3b^2|$.
   (a) Prove that the following map is a ring homomorphism
   $$\pi : \mathbb{Z}[\sqrt{3}] \longrightarrow \mathbb{Z}/11, \qquad a + b\sqrt{3} \longmapsto a + 5b$$
   (b) Prove that $\mathbb{Z}[\sqrt{3}]$ with the function N forms a Euclidean domain.
   (c) Prove that if $N(a + b\sqrt{3})$ is a prime integer, then the principal ideal $(a + b\sqrt{3})$ is a prime ideal of $\mathbb{Z}[\sqrt{3}]$. (**Hint.** You may use the conclusion of part (b).)

4. (10 points) Let $A = \mathbb{Z}$ be the ring of integers. Consider the subset
   $$S = \{70n + 1 \mid n \in \mathbb{Z}\}.$$
   (a) Prove that S is a multiplicative subset of $\mathbb{Z}$.
   (b) Find all the maximal ideals of the ring $S^{-1}A$.

5. (10 points) Let a and b be positive integers. Let $c > 0$ be the greatest common divisor of a and b. Prove that $x^c - 1$ is the greatest common divisor of the polynomials $x^a - 1$ and $x^b - 1$ in the polynomial ring $\mathbb{Q}[x]$.

6. (10 points) Let B be an integral domain and let A be a subring of B. Let M be a projective A-module. Prove that the tensor product $B \otimes_A M$ is a projective B-module.

MICHIGAN STATE
UNIVERSITY

Algebra Qualifying Exam

January 2019

Instructions:

- Read each problem carefully.
- Write legibly and in complete sentences.
- Cross off anything you do not wish graded.
- This exam has 9 pages with 6 questions, for a total of 70 points. Make sure that all pages are included.
- You may not use books, notes, or electronic devices.
- You may ask proctors questions to clarify problems on the exam.
- Graded work should only be written in the provided spaces (not on scratch paper) unless otherwise approved by a proctor.
- You have 120 minutes to complete this exam.

Good luck!

1. (10 points) Let G be a group of order $175 = 5^2 \cdot 7$. Prove that G is an abelian group.

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Define $N(a + b\sqrt{3}) = |a^2 - 3b^2|$.

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$$\pi : \mathbb{Z}[\sqrt{3}] \longrightarrow \mathbb{Z}/11, \qquad a + b\sqrt{3} \longmapsto a + 5b$$

(b) Prove that $\mathbb{Z}[\sqrt{3}]$ with the function N forms a Euclidean domain.

(c) Prove that if $N(a + b\sqrt{3})$ is a prime integer, then the principal ideal $\left(a + b\sqrt{3}\right)$ is a prime ideal of $\mathbb{Z}[\sqrt{3}]$.

Hint. You may use the conclusion of part (b).

4. (10 points) Let $A = \mathbb{Z}$ be the ring of integers. Consider the subset

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(a) Prove that S is a multiplicative subset of $\mathbb{Z}$.

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